DOCUMENT RESUME

ED 285 767 SE 048 355

AUTHOR Clifford, Paul C.; And Others
TITLE Sets, Probability and Statistics: The Mathematics of Life Insurance.
INSTITUTION American Council of Life Insurance, Washington, D.C. Education and Community Services.
PUB DATE Sep 78
NOTE 39p.; For a computer program based on this text, see SE 048 356.
AVAILABLE FROM American Council of Life Insurance, Company Services, 1001 Pennsylvania Avenue, NW, Washington, DC 20004-2599 (25 copies to teachers free, then $.15 each; order #176).
PUB TYPE Guides - Classroom Use - Materials (For Learner) (051) -- Tests/Evaluation Instruments (160)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Insurance; Mathematical Concepts; Mathematical Enrichment; Mathematics Education; *Mathematics Instruction; Mathematics Tests; *Probability; School Business Relationship; Secondary Education; *Secondary School Mathematics; *Set Theory; *Statistics

ABSTRACT

The practical use of such concepts as sets, probability and statistics are considered by many to be vital and necessary to our everyday life. This student manual is intended to familiarize students with these concepts and to provide practice using real life examples. It also attempts to illustrate how the insurance industry uses such mathematic concepts for everyday business activities. The booklet was designed to be used in basic and advanced high school mathematics classes. The eight lessons contain activities involving: (1) sets; (2) sample spaces; (3) probability; (4) statistics; (5) mortality tables and life insurance; (6) interest; and (7) insurance premiums. (TW)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

Sets, probability and statistics

ED 285767

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

X This document has been reproduced as received from the person or organization originating it.

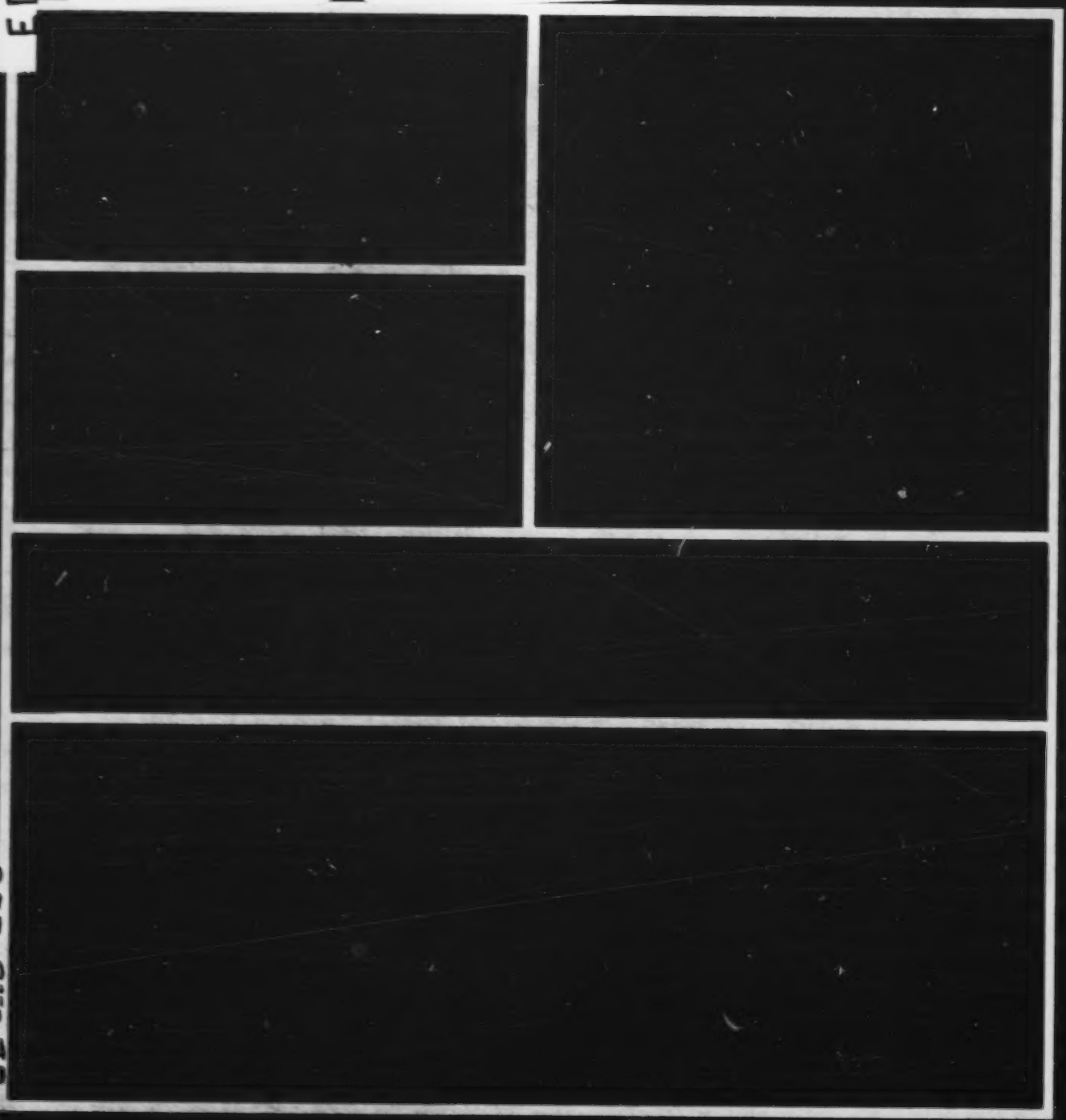
☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY

I M Goldfenbein

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."



SE 048 355

Sets, probability and statistics:
The Mathematics of Life Insurance

prepared in cooperation with

Paul C. Clifford, former Chairman
Department of Mathematics
Montclair State College
Montclair, New Jersey

Mildred Keiffer
Former Supervisor of Mathematics
Cincinnati Board of Education
Cincinnati, Ohio

Max A. Sobel
Professor of Mathematics
Montclair State College
Montclair, New Jersey

Education and Community Services, American Council of Life Insurance, 1850 K Street, N.W., Washington, D.C. 20006

Sets, Probability and Statistics

the mathematics of life insurance

Table of contents

Sets	3
Sample Spaces	7
Probability	8
Statistics	16
The mortality table and life insurance	21
The matter of interest	25
Figuring the premium	28
The uses of life insurance	31

Introduction

sets . . .
probability . . .
statistics . . .

These are common words. You have heard them used in ordinary conversations and in radio and television broadcasts, and you read them in newspapers and magazines. Advertisements urge you to buy a set of books or dishes. The probability of rain tomorrow is reported, and motorists are warned about the number of expected traffic fatalities during holiday weekends. Statistics from last night's ball game or the current cost of consumer goods are analyzed.

Sets, probability and statistics are mathematical concepts. They are invaluable working tools in the worlds of business, industry, science and engineering. The wage-earner balancing the family budget relies upon them. Even children are concerned with sets and probability, although they may not realize it, when they play games or choose teams.

The insurance business is based in large measure on the practical application of the concepts of sets, laws of probability and the use of statistics. Insurance of all types — life, health, pensions, fire, automobile, casualty, property — relies on these concepts. In the life insurance business, for instance, the money paid to purchase life insurance protection is called a "premium." The life insurance company computes the premium on the basis of the calculations explained in this booklet. Health insurance companies calculate the probability of people having a serious illness or accident and statistics show the actual health experience of various sets of people.

Mathematics and the practical use of such concepts as sets, probability and statistics are as vital and necessary to the life and health insurance business as they are to the construction of bridges and to space exploration. The results of the practical application of these concepts surround you each day.

This booklet will explain and apply the mathematical concepts of sets, probability and statistics.

Bob has a *set* of golf clubs. Dad bought a new *set* of tires. We unpacked a *set* of dishes. Each time we use the term "set" in these sentences, we are expressing a mathematical concept that is useful in understanding arithmetic, algebra and geometry. In the next few pages you will examine this concept, and learn how useful it can be in mathematics.

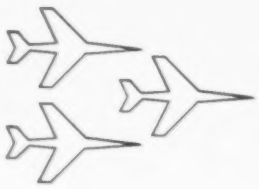
A set is simply a collection of things of the same kind: for example, the dishes or the golf clubs. Or we may speak of the

set of days of the week, or
set of months in the year, or
set of beautiful girls with blond hair, or
set of teachers of mathematics, or
set of integers from 1 through 99, or
set of numerals on the six faces of a die, or
set of 16-year olds alive today who will live to be 100.

Each item in a set is called an *element* or *member* of the set. Monday is a member of the set of days in the week; golden-haired Sue is a member of the set of blondes; 31 is an element in the set of integers.

There are many words we use in everyday language that have the same meaning as set. For example, each of the *italicized* words below denotes a collection:

herd of cattle
brace of pheasants
flock of sheep
gaggle of geese
squadron of planes



How many more words can you find that have the same meaning as set?

In mathematics, we usually are concerned with sets of numbers. The set of integers from 1 through 9, for instance, denotes a collection of certain numbers. Such a collection is displayed by using a pair of braces { } to represent the set, and listing the elements within the braces. To identify the set, we often arbitrarily assign a capital letter. Thus, we may represent the set of integers from 1 through 9 as follows:

I = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

Here are other examples:

The set of integers between 3 and 8:	**R = {4, 5, 6, 7}.**
The set of integers from 1 through 5:	**T = {1, 2, 3, 4, 5}.**
The set of numerals that appear on the six faces of a die:	**D = {1, 2, 3, 4, 5, 6}.**
The set of words used to distinguish the two faces of a coin:	**C = {heads, tails}.**

Now, consider the following sets:

The set of integers between 2 and 3.
The set of even numbers between 6 and 8.
The set of mathematics teachers who have visited Mars.
The set of ninth grade students who are astronauts.
The set of triangular faces on a cube.

For each of these you see that the set has no members, and therefore we call it the *empty set* or the *null set*, and use the symbol ϕ to indicate such a set. See if you can give five different examples of the empty set.

problems:

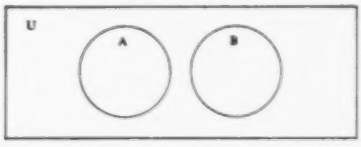
List the elements in each of the following sets:

1. **The set of integers from 2 through 8.**
2. **The set of integers between 7 and 12.**
3. **The set of integers greater than 5 but less than 9.**
4. **The set of even integers greater than 0 but less than 12.**
5. **The set of fractions whose numerator is 1 and whose denominator is an integer between 1 and 10.**
6. **A cube has the first six letters of the alphabet on its faces. List the set of possible outcomes if the cube is tossed.**
7. **A box contains five slips of paper on which are printed the vowels of the alphabet, one vowel to each slip. List the set of possible outcomes if a slip is drawn from the box.**

Venn diagrams

The English mathematician, John Venn, introduced the idea of using diagrams to represent relationships between sets. These are now called *Venn diagrams* and are useful in solving problems in a variety of different areas of mathematics. He represented the entire set of elements under a given discussion as the set of points within a rectangle. Then he used the set of points within circles to represent subsets.

In the following diagram, U represents the set of all integers from 1 through 50. A will represent the set of integers that are multiples of 5; B represents the set of integers that are multiples of 11. The remaining members in U would be all integers not included in sets A or B.



operations with sets

There are many different relationships between sets that are of interest. We shall explore several of these relationships.

intersection of sets:

Consider the following two sets:

$A = \{1, 2, 3, 4, 5, 6, 7\}$.
$B = \{3, 5, 7, 9, 11\}$.

By the *intersection* of these two sets we mean the collection of elements that are found in each of them, which in this case is a set consisting of 3, 5 and 7. Thus we would say that the intersection of sets A and B is the set C:

$C = \{3, 5, 7\}$.

To represent the word "intersection", we use this symbol: $\cap$.
$A \cap B$ is read "A intersect B". Set C, above, would correctly be written: $A \cap B = \{3, 5, 7\}$.

Here are several additional examples:

1. $R = \{2, 3, 4, 5, 6\}$.
   $S = \{5, 6, 7, 8, 9\}$. $R \cap S = \{5, 6\}$.

2. $M = \{2, 4, 6, 8, 10, 12\}$.
   $N = \{4, 6, 8\}$. $M \cap N = \{4, 6, 8\}$.

3. $X = \{1, 2, 3, 4, 5\}$. $X \cap Y = \phi$. Why?
   $Y = \{6, 7, 8, 9, 10\}$.
4. $P = \{*, !, ?\}$. $P \cap T = \{?\}$.
   $T = \{?, \$, ¢, \%\}$.

union of sets:

Consider the following two sets:

$A = \{1, 2, 3, 4\}$.
$B = \{3, 4, 5, 6, 7\}$.

By the *union* of these two sets we mean the collection of elements that are members of at least one of the sets. That is, the union of the sets A and B is the set C:

$C = \{1, 2, 3, 4, 5, 6, 7\}$.

Note that we listed the elements that appeared either in set A or in set B. Each element is listed only once although it may happen to appear in both sets. The symbol which represents union is $\cup$.

$A \cup B$ is read "A union B".

Set C in our example is $A \cup B$ and $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

Here are additional examples:

1. $K = \{2, 5, 7\}$. $K \cup L = \{1, 2, 3, 5, 6, 7\}$.
   $L = \{1, 3, 5, 6\}$.
2. $M = \{0, 1, 2, 3, 4, 5\}$. $M \cup N = \{0, 1, 2, 3, 4, 5\}$
   $N = \{2, 3, 4, 5\}$.
3. $X = \{1, 2, 3, 4\}$. $X \cup Y = \{1, 2, 3, 4, 5, 6, 7\}$.
   $Y = \{5, 6, 7\}$.

complement of a set:

Consider the following two sets:

$U = \{1, 2, 3, 4, 5\}$.
$B = \{1, 3, 5\}$.

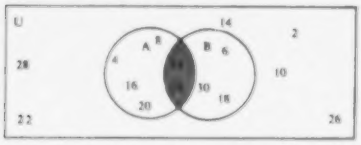
By the *complement* of the set B we mean the set of elements in U that are not in B. Thus, the complement of B is the set $\{2, 4\}$. We write this as follows: $\overline{B} = \{2, 4\}$ and we read it as "the complement of B is the set $\{2, 4\}$".

An example: if U represents the set of 15-year olds alive today, and R is the set of these 15-year olds who will still be living in the year 2000, then $\overline{R}$ is the set of these 15-year olds who will not be alive in the year 2000.

Another example: let U be the set of numerals that appear on the faces of a die. Let T be the set of numerals on a die that are less than 5. What is the complement of T?

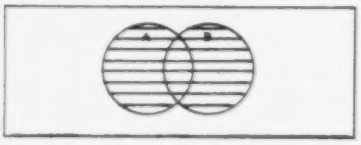
$U = \{1, 2, 3, 4, 5, 6\}$.
$T = \{1, 2, 3, 4\}$.
$\overline{T} = \{5, 6\}$ that is, the set of elements in set U that are not in set T; in other words, the set of elements that are not less than 5.

In the diagram below, we illustrate the *intersection* of sets and this time we use the following sets:



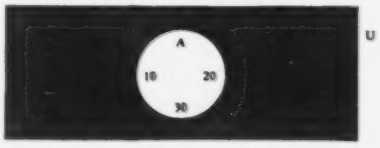
$U = \{2, 4, 6, 8, 10, 12 \ldots 30\}$.
$A = \{4, 8, 12, 16, 20, 24\}$.
$B = \{6, 12, 18, 24, 30\}$.
$A \cap B = \{12, 24\}$.

We can picture the *union* of two sets with this diagram:



$A = \{2, 3, 4, 6, 7, 8\}$.
$B = \{1, 3, 5, 7\}$.
$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

The *complement* of a set may be diagrammed in the following way:



$U = \{5, 10, 15, 20, 25, 30, 35\}$.
$\underline{A} = \{10, 20, 30\}$.
$\overline{A} = \{5, 15, 25, 35\}$.

problems:

For each of the following pairs of sets, list the elements in

(a) their intersection and (b) their union:

1. $A = \{1, 2, 3, 4, 5\}$; $B = \{2, 4, 6, 7\}$.
2. $M = \{1, 3, 5, 7\}$; $N = \{2, 4, 6, 8\}$.
3. $K = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $L = \{3, 4, 5, 6\}$.
4. $S = \{5, 10, 15, 20, 25\}$; $T = \{10, 20, 30, 40\}$.
5. $P = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$; $Q = \left\{\frac{1}{3}, \frac{1}{7}, \frac{1}{9}\right\}$

	Boys	Girls
A students	5	8
Not A students	8	9

6. The chart represents the distribution of grades in a 10th grade mathematics class:

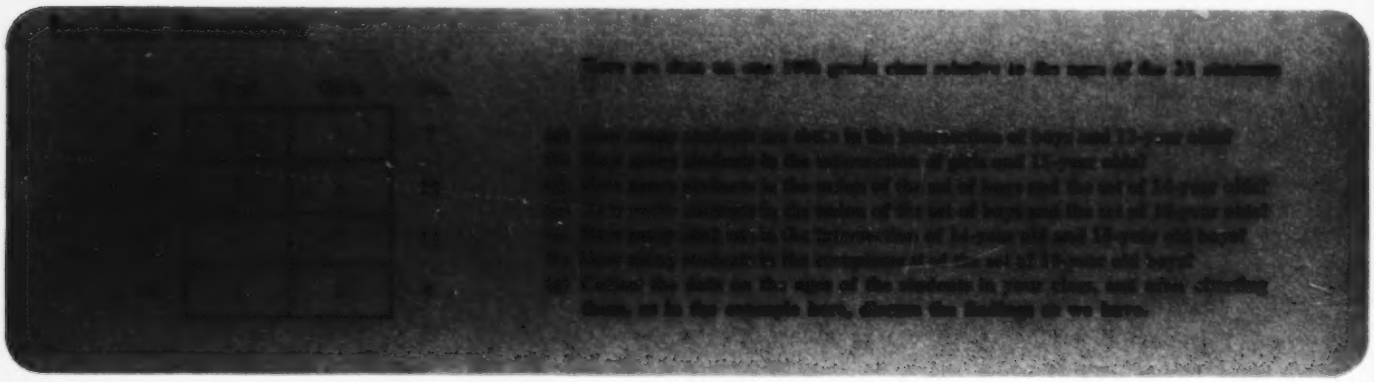
What is the number of students in the set of boys who got A?

What is the number in the set of girls who did not get A?

Assume that all the girls were excused from class and all the A students were also excused. How many were excused? (Do you see that this represents the union of the set of girls and the set of A students?)

7. Let $U = \{1, 2, 3, 4, 5, 6\}$. Find $\overline{A}$, where set A is defined as follows:
   - (a) $A = \{1, 3, 5\}$.
   - (b) $A = \{1, 3, 5, 6\}$.
   - (c) $A = \{1, 2, 3, 4, 5, 6\}$.
   - (d) $A = \phi$.
8. Let $A = \{1, 2, 3, 4, 5\}$; $B = \{2, 5, 7, 8\}$; $C = \{1, 5, 8, 9\}$.

   Find:
   - (a) $A \cup B \cup C$
   - (b) $A \cap B \cap C$
   - (c) $(A \cup B) \cap C$
   - (d) $(A \cap B) \cup C$



Another kind of set that is very important in mathematics is one that consists of a collection of all the possible outcomes of an event. Suppose, for example, you are asked to toss two coins, a dime and a quarter. In how many ways can they land? Here is a list of all the possible outcomes of the toss:

Dime	*Quarter*	
H	H	Both heads
H	T	Dime, heads; quarter, tails
T	H	Dime, tails; quarter, heads
T	T	Both tails

The listing of all the possible outcomes of an experiment is called the *sample space*, which is the set of possible results. The sample space for the above experiment may be written

{HH, HT, TH, TT}.

In this notation, the first letter in each pair represents the outcome of the toss of the dime, the second letter represents the quarter.

Consider the experiment when three different coins are tossed. The list of all possible outcomes appears as follows:

Dime	*Quarter*	*Penny*
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

Again, we can list the set of all possible outcomes as follows:

{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

This is a sample space for three coins. Note that there are 8 different possible outcomes. How many of these produce three heads? How many produce three tails?

Suppose you are asked to toss a die. In how many different ways can the die land face up? Assuming that you toss a legitimate die, it can land with any one of six faces up and thus a sample space for this experiment is the set:

{1, 2, 3, 4, 5, 6}.

Now, consider the more difficult problem of tossing a pair of dice. To help keep these clear in our minds, let us agree that one die is blue and the other is green. Thus an outcome of 2 on the blue die and 5 on the green die is different from an outcome of 5 on the blue die and 2 on the green die. We can set up a sample space for this experiment by means of the table at right where we list the outcomes on the blue die along the left

Outcomes of green die

Outcomes of blue die	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

side, and those of the green die across the top. Then the entries of the table are to be interpreted as follows:

(2, 5): 2 on blue die, 5 on green die.

(5, 2): 5 on blue die, 2 on green die.

problems:

List a sample space for each of the following experiments:

1. **A die is tossed, then a coin is tossed.**
2. **A three-digit numeral is to be formed from the digits 2, 4 and 5 with no digit to be used more than once in a numeral.**
3. **A box contains four slips of paper, each with a letter a, b, c, or d written on it. You are to draw one slip of paper from the box, replace it, and then draw a second slip.**
4. **Repeat exercise 3, except this time you are to draw 2 slips of paper from the box, one at a time, without replacement.**
5. **Four coins are tossed.**

Using the sample space for the green and blue dice on page 7, answer these questions:

6. **How many of the outcomes show both of the dice with the same outcome? For example, (1, 1), etc.**
7. **How many of the outcomes have a sum of 7?**
8. **List the outcomes where the blue die is 1.**
9. **List the outcomes where the green die is 1.**
10. **Using the results of problems 8 and 9, list the *intersection* and the *union* of the two sets of outcomes.**

Probability chapter three

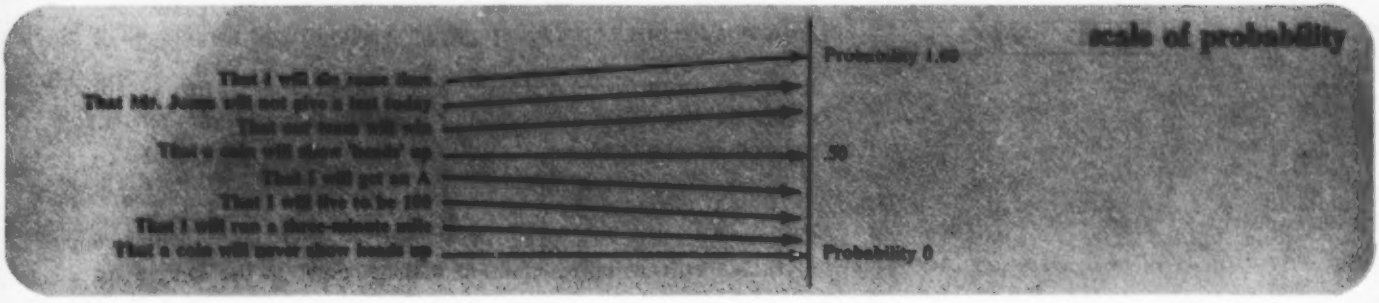
The word "chance" is one we frequently use in conversation. Like many words it has a variety of meanings. We may say that we took a *chance* crossing the street; or that we took a *chance* on a raffle. We discuss the *chances* that the Giants will win the pennant, that our next missile will hit the moon. We wonder what the *chances* are that we will get an A on our next exam; or that we will live to be 100. Usually in such discussions we are content with rather rough approximations. We may say "I haven't a chance of getting an A"; or "I'm sure Mr. Jones wouldn't give us a test today."

Sometimes we get more quantitative. We say "I'll bet you two to one that our team wins"; or "The chances are 50-50 that this coin will come up heads"; or "There is a bare mathematical possibility that our team could

win the tournament." When the notion of chance is applied in so many ways, it is not surprising that we must exercise considerable care in giving it a definite mathematical meaning. This is what we must do when we define *probability*. In the following pages we will consider probability and learn how mathematicians can measure it.

Let us consider a suitable scale for probability. Look at the chart below, which shows a scale starting at 0 and extending to 1, with .50 marking the half-way point. The top end represents an event we know with certainty will happen. The bottom of the scale represents an event that is absolutely improbable. Thus, the probability of always getting a sum of 7 in a toss of two ordinary dice is .50. The probability that you will run a three-minute mile is recorded as all but impossible; this, however, is a physical probability as opposed to a logical improbability. (In the 1930's there were many articles written "proving" that a four-minute mile was "impossible".) Some events are easy to place on the scale, other events cause us to wonder.

We need to determine how we can arrive at a measure of the probability of a real event. There are two quite distinct ways of doing this and we shall consider each of them in turn.



equally likely events

Consider the experiment of tossing a coin. There are two possible outcomes: heads or tails. The sample space for this experiment consists of just these two outcomes: {H, T}.

It seems reasonable to assume that each of these two events is *equally likely to happen*. Based on this assumption, we state that the probability of heads is $\frac{1}{2}$ and that the probability of tails is $\frac{1}{2}$, and we would write it as follows: $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$.

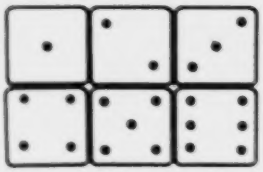
$$\mathbf{P\,(H) = \tfrac{1}{2},\ P\,(T) = \tfrac{1}{2}.}$$

It is hard to say whether we reach this conclusion by examination of the coin or by experience. Coin-tossing is a very old human activity and most of us tossed coins long before we ever reached the stage of talking about probability.

We have considerable evidence to show that when a coin is tossed a large number of times, the fraction of heads does approximate $\frac{1}{2}$.

During World War II, one mathematician in an internment camp tossed a coin 1,000 times and then repeated this experiment over and over. He kept a record of the results. In ten sets of 1,000 tosses, he found that the number of heads was: 502, 511, 497, 529, 504, 476, 507, 528, 504 and 529. Notice how the numbers of heads clustered around 500 (one-half of 1,000) – although none of the numbers was precisely 500. It is important to examine the results of such experiments to find if they do agree with our assumptions.

Consider tossing a die, the six faces of which are marked by dots as shown.



Most people find it reasonable to assume that when a die is tossed, one face is as likely to turn up as another. Thus, to each of the six possible outcomes we assign the same probability, $\frac{1}{6}$. (Frequently people say that a die is 'fair' or 'unbiased'; they mean that the imperfections in the die are minor and that we are to assume that each face has the probability $\frac{1}{6}$).

If we consider the six possible outcomes of tossing a die, there are various *events* that we might consider. For instance, there are six elementary *events*, indicated by the set:

$$\{1, 2, 3, 4, 5, 6\}.$$

The above set lists what we call the elementary events, since each outcome of the experiment corresponds to a single point in the sample space. However, it might be that we are only concerned as to whether the outcome is an *even* number or an *odd* number. Thus the event: OUTCOME IS EVEN consists of the three elementary events $\{2, 4, 6\}$. Since three of the six equally likely events result in even numbers, we say that the probability that the outcome is even is $\frac{3}{6}$.

problems:

1. **In a bag there are 6 white balls and 1 red ball. What is the probability of picking a red ball on the first draw?**
2. **In a bag there are 6 white balls and 2 red balls. What is the probability that you will pick a white ball on the first draw?**

3. What is the probability that you will get a 3 when you toss a die?

4. What is the probability that you will not get a 3 when you toss a die?

5. A pupil is required to read a play, a short story, and a poem. What is the probability that he reads the poem first? (Assuming that each is equally likely to be in the book he picked up first.)

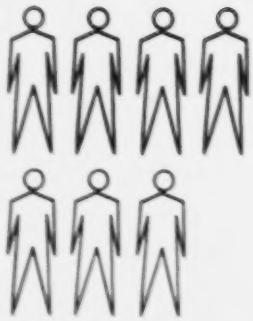
6. If the batting average of the star shortstop is .289, what is the probability that he will get a hit the next time he is at bat?

7. If you have 1 dime and 7 pennies in your pocket, what is the probability that the coin you take out at random will be a penny? a dime?

complementary events

Suppose you enter a classroom in which there are seven students, none of whom you know. Four of the students are in the 10th grade and three are in the 11th grade.

$S = \{10, 10, 10, 10, 11, 11, 11\}$.



From this set of seven students, what is the probability that you will select a 10th grader on your first try? If A represents the event that you select a 10th grader

$P(A) = \frac{4}{7}$. Why?

Now we can see that the probability that you will select a student from the 11th grade is

$P(\bar{A}) = \frac{3}{7}$.

This notation designates the probability of the complementary event. It is simply the ratio of the events which are not in A to the total number of events in set S.

You recall that in Chapter One we used the notation $\bar{A}$ to denote the complement of set A. We have used it here in a similar manner. Since m of the elementary events correspond to event $\bar{A}$, the balance, or n-m, of these events corresponds to event $\bar{A}$. To determine the probability of an event A, we would use this formula:

$$P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} \quad \text{or } P(\bar{A}) = 1 - P(A)$$

more on sample spaces

Recall the experiment in which we tossed two coins, a dime and a quarter. How did we list the possible outcomes? In other words, what was the sample space? Suppose we consider again the following set of outcomes, which shows how each coin might fall.

Outcome	*Dime*	*Quarter*
1	H	H
2	H	T
3	T	H
4	T	T

$S = \{HH, HT, TH, TT\}$.

Assuming these four outcomes are equally likely, we assign probability $\frac{1}{4}$ to each.

$$P(HH) = \tfrac{1}{4}$$
$$P(HT) = \tfrac{1}{4}$$
$$P(TH) = \tfrac{1}{4}$$
$$P(TT) = \tfrac{1}{4}$$

$$P(HT \cup TH) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

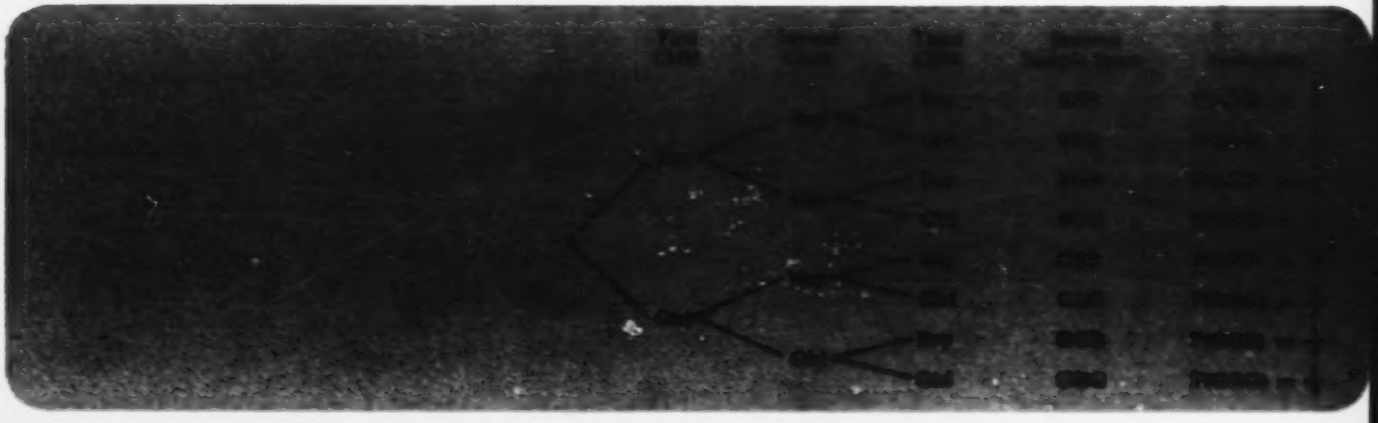
In this experiment, we could have recorded the results differently. At each toss we might have recorded only the number of heads occurring. The set of outcomes would have been $S_1 = \{2, 1, 0\}$. Or we might have been interested in whether the outcomes were alike (a) or different (d). Then the set of outcomes would have been $S_2 = \{a, d\}$.

Each of these sets provides a list which includes all possible outcomes of the experiment. Each set is a sample space.

As another example, consider all the three-child families in your school district. In your notation, "B" will stand for "boy" and "G" for "girl". Use a triple set of letters for each family, with the oldest, the middle and the youngest child recorded in that order. Thus, BBG means that the first two children are boys and the youngest is a girl. A sample space would be:

{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}.

Another way of showing this is with a *tree diagram:*



Assuming these eight outcomes are equally likely, we have assigned the probability $\frac{1}{8}$ to each.

Another sample space might be that which lists merely the number of boys in each family: $N = \{3, 2, 1, 0\}$. Since only one event in the ordered

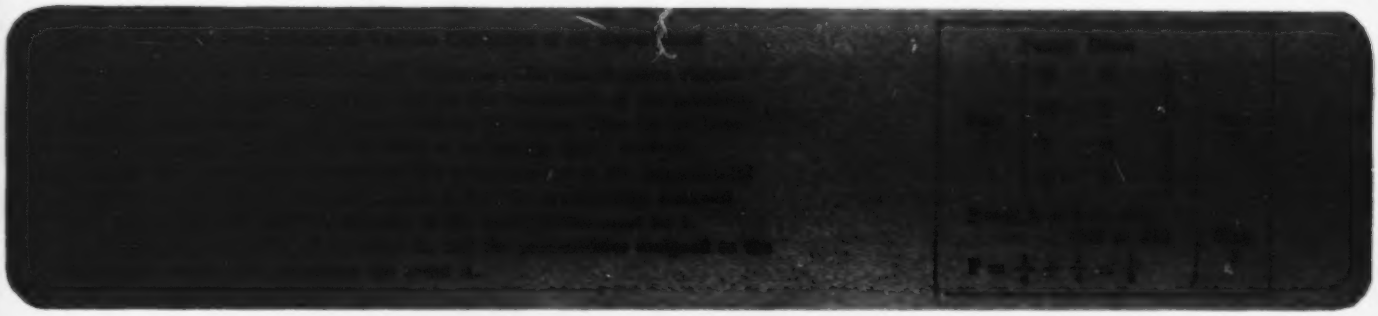
sample space corresponds to three boys, we would assign

$$P(3 \text{ boys}) = \frac{1}{8}.$$

However, there are three events in the ordered sample space that correspond to the event Two Boys (BBG, BGB, GBB). Thus we would have $P(2 \text{ boys}) = \frac{3}{8}$. Complete the analysis and show that

$$P(3 \text{ boys}) = \frac{1}{8}$$
$$P(2 \text{ boys}) = \frac{3}{8}$$
$$P(1 \text{ boy}) = \frac{3}{8}$$
$$P(0 \text{ boy}) = \frac{1}{8}$$

You might check your intuition by noticing the relationship between this problem and that of tossing three coins. That is the event HHH on three coins might be considered equivalent to BBB in the three-child family.



probability of A or B: P (A$\cup$B)

While it is possible to compute probabilities directly from elementary outcomes, as we have been doing, it is frequently easier to compute them from other events. For example, we have a bowl of 100 beads of which 10 are blue, 20 are red, and 70 are white. What is the probability of drawing a blue bead? We know that the result of the draw will be either a blue, red, or white bead – it cannot be a foreign color, or two colors; nor can a bead of each color happen to be drawn at one time. When such a condition exists – when two or more events cannot possibly occur at once, we say we have *mutually exclusive events.*

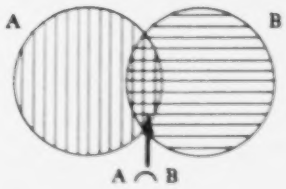
The sets of beads are mutually exclusive. We would assign probabilities as follows:

$$P(B) = \frac{10}{100} = .10$$
$$P(R) = \frac{20}{100} = .20$$
$$P(W) = \frac{70}{100} = .70$$

What is the probability of drawing a blue or a red bead? Since the set B and the set R are mutually exclusive, $P(B \cup R) = P(B) + P(R)$. Thus $P(B \cup R) = .10 + .20 = .30$.

Similarly, if A, B and C are mutually exclusive events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 1$.

Outcomes of blue die	Outcomes of green die: 1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)



Sometimes we want to know the probability that A *or* B will occur; where A and B are *not* mutually exclusive. For example, we may ask, on the toss of two dice, for the probability that at least one of the dice show a 2. Now if we refer to the diagram on page 7 , we can see that there are six elementary events where the green die shows a 2, and also six events where the blue die shows a 2. The general pattern of the sets G (green die shows a 2) or B (blue die shows a 2) is shown at left.

We see that these two sets intersect, and that their intersection is the event (2, 2). The probability of this event has been counted in both Set G and in Set B, and to correct for this we must subtract the probability of the intersection ($G \cap B$):

$$P \text{ (at least one 2)} = \tfrac{6}{36} + \tfrac{6}{36} - \tfrac{1}{36} = \tfrac{11}{36}.$$

This illustrates the general theorem for finding the probability of A or B.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

The mutually exclusive case is a special one when $P(A \cap B) = 0$.

probability of A and B: P (A∩B)

If the events A and B are independent, we can compute the probability of the event ($A \cap B$) from the separate probabilities P(A) and P(B).

$$P(A \cap B) = P(A) \times P(B).$$

Actually, this is a definition of independence of probability. We have already encountered several examples where it could have been used. We found that the probability of obtaining heads on tossing a single coin was $\frac{1}{2}$. From this we compute the probability of obtaining two heads on tossing two coins:

$$P(H \cap H) = P(H) \times P(H) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}.$$

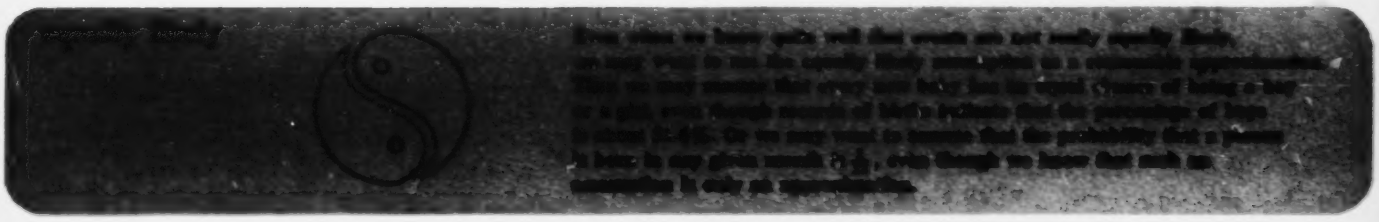
This agrees with the result we had on page 12.

Also, for the three-child family we found that the probability of a three-boy family was $\frac{1}{8}$. Assuming that the probability of a boy is $\frac{1}{2}$, and that the sex of the first child has no effect on the sex of the next, we would have

$$P(3 \text{ boys}) = P(B \cap B \cap B) = P(B) \times P(B) \times P(B) = \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{8}.$$

As one more example, consider the probability that both the green die and the blue die, in the chart on this page, show 2's. We know that $P(G = 2)$ is $\frac{1}{6}$, $P(B = 2) = \frac{1}{6}$. Since we feel that the green die outcome cannot affect the outcome of the blue die, we can assume independence. Thus

$$P(G = 2 \cap B = 2) = P(G = 2) \times P(B = 2) = \tfrac{1}{6} \times \tfrac{1}{6} = \tfrac{1}{36}.$$



problems:

1. The chocolates in a box of candy are the same size and appearance. However, they differ as to the center of the candy. While the distribution of any type in the box is random, the number of each type is as follows:

Nut		Fruit	
10	Peanut	7	Cherry
8	Almond	5	Raspberry
12	Walnut	4	Strawberry
		4	Lemon

Find the following probabilities:

(a) P(Nut Center) = P(N) =

(b) P(Not Lemon) = $P(\overline{L})$ =

(c) P(Berry Center) = $P(R \cup S)$ =

2. Look again at the chart on page 6 showing the distribution by age of the members of a 10th grade class. If a student is selected at random, what is the probability that:

(a) the student is age 13

(b) the student is a boy

(c) the student is both a boy and age 13

(d) the student is a boy or age 13

(e) the student is not age 13

(f) the student is a boy, given that the student is 16.

3. What is the probability on the third toss of a coin that it will turn up heads? On the hundredth toss?

4. What is the probability that you will get a 3 or a 4 when you toss a die?

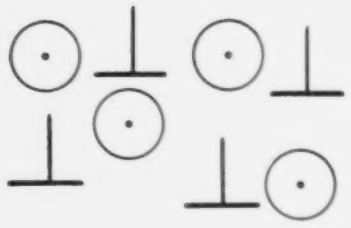
5. What is the probability that you will not get a 3 or 4 when you toss a die?

6. If your failing rate is one test in 4, what is the probability that you will pass the next test?

7. If statistics show that 375 sets of triplets were born in 4,203,000 births, state the probability that a set of triplets will occur as a ratio with 1,000 in the denominator.

8. Collected data showed that in one year, in a total of 94,780 accidental deaths, 39,628 resulted from motor vehicle accidents. State the probability (based on these figures) that an accidental death resulted from a motor vehicle accident. State this probability as a rate per 1,000.

Statistics chapter four

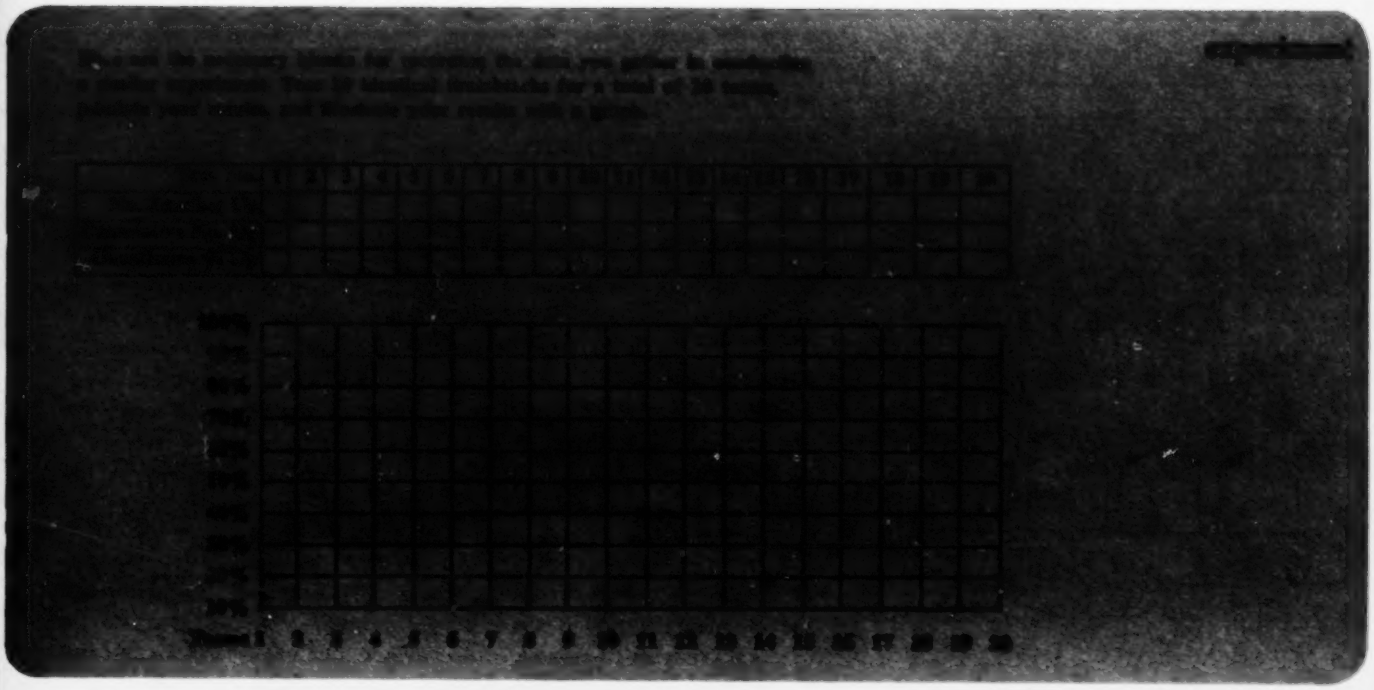


Look again at the scale of probability on page 9 . This scale shows that the probability of getting an A is less than $\frac{1}{2}$. And the probability that one will live to be 100 is almost at the 0 mark. We may ask on what basis these probabilities were determined.

How would you determine the probability that you will drive a car for the next 12 months without an accident? Or what is the probability that your school building will not burn down this year? Or what is the probability that the Yankees will win the pennant next season?

These are rather different problems from those we completed in the previous chapter, but they are the kinds of problems that are important to home owners, to auto drivers, to insurance companies, and to sports fans. These are problems in which we arrive at an expected outcome as to what will happen *based on data we collect from a sample*. Because the probability that something will happen or will not happen is unknown, we start by observing data that we gather or that has already been collected for us.

Example: We can't tell how thumbtacks will land when we toss them until we try. We could toss 10 thumbtacks for a total of 20 tosses and record the number landing UP as shown in the accompanying table. We can also compute the per cent that land UP on each toss. We can keep a cumulative score of the number landing UP and then compute the cumulative per cent landing UP. Finally, we can graph that cumulative per cent, as is done in the graph below the table.



What is the probability, based on your findings, that a thumbtack will land UP?

In the thumbtack experiment, you supplied your own data based upon the results of an experiment. Very often we can use data already compiled to determine the probability of an event.

problems:

1. The Madison High School baseball team has played 15 games this season.

The record for three of the players is as follows:

Player	Times At Bat	Hits
Centerfielder	60	27
Shortstop	50	18
1st Baseman	45	14

What is the probability that the centerfielder gets a hit next time at bat?

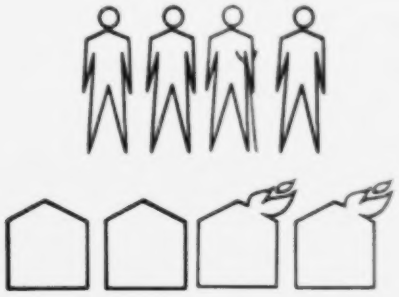
Express your answer as a 3-place decimal.

2. From the following data, what is the probability that a non-fatal accident will have occurred in the home? On the job?

Non-fatal Accidental Injuries, 1971	
Motor Vehicle	2,000,000
In Public Places	2,800,000
At Home	4,200,000
On the Job	2,300,000
Total, non-fatal accidents	11,200,000*

***All injuries do not add to total since some injuries are included in more than one classification.**

3. **A magazine subscription service that solicits business by telephone has records that show that of every 200 telephone calls made, on the average there will be 44 orders of which 39 are actually paid for. What is the probability that a solicitor, on any given call, will receive an order? What is the probability that he will receive an order and that it will be paid for?**



probability and insurance

One of the large businesses in the United States which relies heavily on the theory of probability is insurance. It is very important to an insurance company to know how to measure the risks against which people are buying the insurance. In order to set the premiums, a fire insurance company must have some way of knowing how many fires will occur. An automobile insurance company must be able to predict the number of accidents involving injury, loss of life, and property damage. A life insurance company must know what the expected number of deaths will be in a given group of policyholders.

How then will companies such as these make their estimates? How will they estimate the probability:

(a) that an 18-year old male driver will be involved in an automobile accident this year?
(b) that a new all-brick house in your community will burn?
(c) that a 70-year old man will be hospitalized this year?
(d) that a 16-year old person will die before reaching age 17?

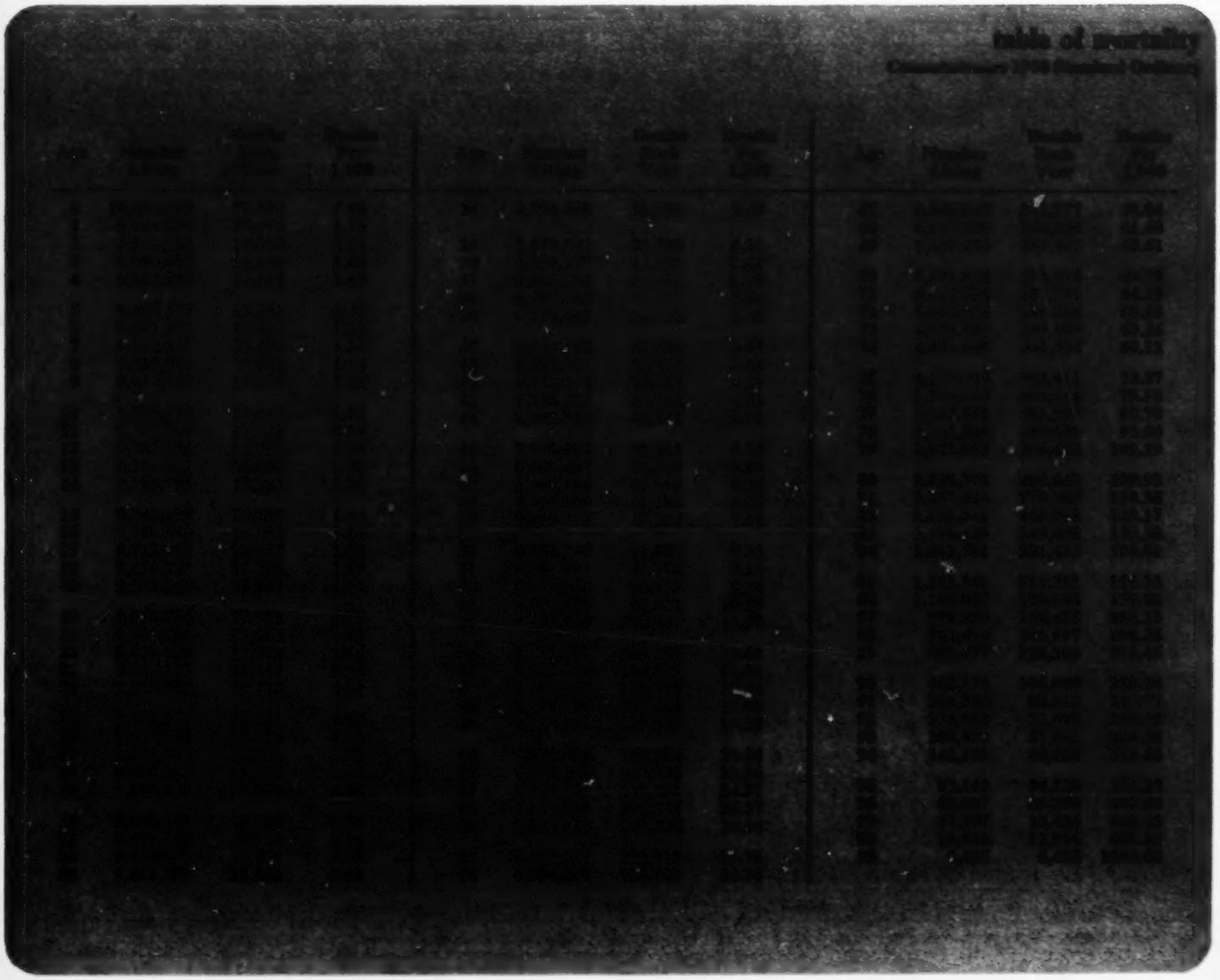
You know immediately that in order to arrive at estimates such as these, some data must be collected. The automobile insurance company will have to gather data on 18-year old male drivers in order to tell what the experience is likely to be – how many accidents this group of drivers has, on the average. Or the fire insurance company must compile statistics on fires among all-brick houses in communities having adequate fire departments. And life insurance companies must have available some statistics that will show how many 16-year olds die, on the average.

Data such as these must be based on large numbers of events. In the thumbtack case, you had to toss a fairly large number of tacks before your experience was adequate to make a prediction. Recall the coin-tossing experiment by the prisoner in the internment camp who tossed several thousand coins and learned that on the average the results were approximately 500 heads out of each 1,000 tosses. Operating in cases such as these is the Law of Large Numbers, which,stated very simply, means that with large groups we can predict fairly accurately what is likely to happen. With a large number of experiments, the ratio of the number of successes to the number of trials does approach the theoretical probability P.

Life insurance companies use sets of statistics called mortality tables to predict how many people of the same age will die in a particular year. The companies assume that what will happen in the future will be similar to what happened in the recent past.

Here is a mortality table that is based on the lives and deaths of policy-holders of several large life insurance companies. This is the Commissioners 1958 Standard Ordinary Table of Mortality which we will refer to as the C.S.O. Table. Although it is not used for premium calculations by any large life insurance company, it is used for making other calculations necessary to life insurance company operations.

This is a table of probabilities, also. At age 5, the probability that death will occur within one year is $\frac{1.35}{1000}$. At age 40 the probability that death will occur within the year is $\frac{3.53}{1000}$. It is customary in the table to speak of the annual rate of death per 1,000 persons rather than to call this the probability of death at a given age.

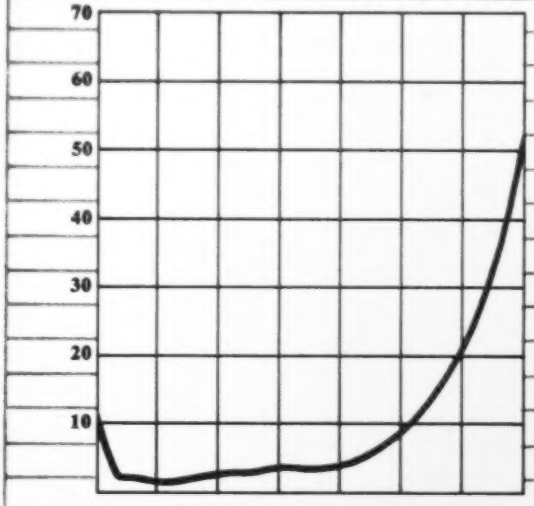


22

problems:

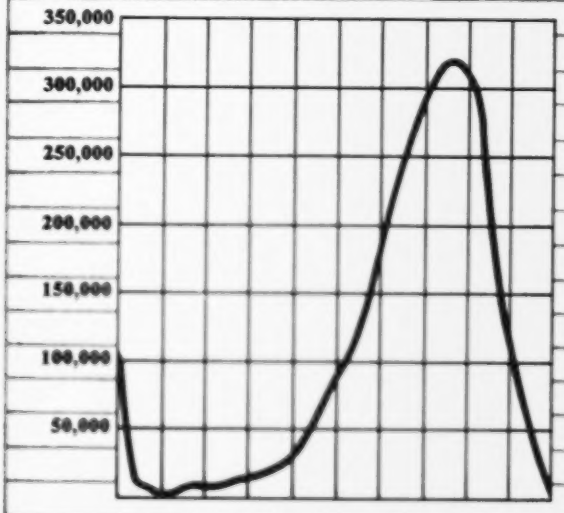
Rate

Graph for Question 6



No. of Deaths

Graph for Question 7



No. of deaths expected at each age, based on 10,000,000 people starting at age 0.

1. From the C.S.O. Table, find the annual death rates per 1,000 and the number of deaths to complete the following:

Age	Death rate per 1,000	No. of deaths
16		
17		
18		
19		
20		

2. (a) From the C.S.O. Table list the set of ages for which the death rate per 1,000 is less than 1.25. List the set of ages for which the death rate is more than 2.00 and less than 3.00.

   (b) Let set A equal the set of ages at which the death rate is more than 1.30; let set B equal the ages at which the death rate is less than 1.95. List the elements in A $\cap$ B.

   (c) Let set A equal the set of ages for which the number of deaths each year is less than 15,000. Let set B equal the set of ages for the number of deaths each year numbering more than 13,000. List the elements in A $\cap$ B.

3. Starting at age 10, plot the death rate at 10-year intervals through age 80 on a graph. Does your graph show the death rate as constant, increasing, or decreasing? Why?

4. In the Table, find the number living at age 10. _____ Find the age in the Table where the number still living is approximately half the number living at age 10. _____ Your answer will represent the age when approximately half the people living at age 10 will still be living. How many years are there between age 10 and the age when half the people are still living? _____ Your answer is what is called the *median future lifetime* at age 10, which simply means the number of years it will take before one half of the 10-year olds will have died.

5. What is the median future lifetime at age 30? _____ What is it at age 49? _____ at age 65? _____ What happens to the median future lifetime as the group advances in age?

6. Here is a graph based on the C.S.O. Table showing the death rate per 1,000. From the graph, what is the death rate per 1,000 at age 10? _____ at age 30? _____ at age 50? _____. At what age does the death rate reach 20? _____

7. Here is a chart showing the numbers of deaths at each age based on the C.S.O. Table. From the graph, how many deaths are expected at age 20? _____ at age 40? _____ at age 70? _____. At what age is the number of deaths greatest? _____ At what age will the number of deaths be the least? _____.

8. Referring again to the C.S.O. Table, how many of the persons who were alive at age 30 were still alive at age 60? _____ What is the probability that a 30-year old person will live to age 60? Give your answer as a common fraction with 1,000 in the denominator.

9. What is the probability that a person who survives to age 75 will continue to survive at least to age 80? Answer to the nearest tenth.

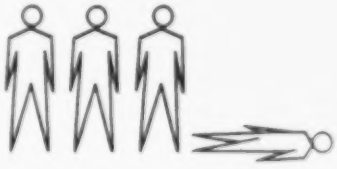
chapter five

The mortality table and life insurance

In the previous chapter you learned that a mortality table is a set of statistics. On the basis of these statistics, a life insurance company can make predictions as to how many policyholders will be expected to die each year, and will thus know how much money will have to be paid out in claims. Although the life insurance company never knows *who* will die, it can tell from the table approximately how many people in a group – all the same age – will die at any given age.

Life insurance companies compile their own mortality tables, based on their own experience with their policyholders, and these tables are revised regularly. It is from these tables that companies secure the basic data needed for establishing the cost of life insurance.

To make up a mortality table the life insurance actuary must determine the death rate of each age. Here, for example, is the actual mortality experience of the Academic Life Insurance Company with its group of 16-year old policyholders:



The number of people age 16 at the start of the year was 5,844. Let S equal the set of 5,844 people alive at age 16. The number who died during the year was 9. Let D equal the set of 16-year old policyholders who died by the end of the year. The probability of dying was $\frac{9}{5,844}$. But in life insurance, instead of stating probability as a fraction, we state it as the death rate per 1,000.

To figure the annual death rate per 1,000, the actuary uses this proportion:

$$\frac{X}{1,000} = \frac{\text{No. dying during year}}{\text{No. living at start of year}}$$

$$X = \text{Death rate per } 1,000$$

With this formula, we can determine X for the 16-year old group:

$$\frac{X}{1,000} = \frac{9}{5,844}$$

$$5,844 \times X = 9 \times 1,000$$

$$X = \frac{9,000}{5,844}$$

$$X = 1.54, \text{ death rate per } 1,000$$

With this calculation, we have learned that out of every set of 1,000 16-year old policyholders of the Academic Life, it is expected that on the average 1.54 will die before reaching age 17.

At age 17, there were 5,835 persons living at the start of the year, and the Company had added 263 new 17-year old policyholders, making a total of 6,098. During the year 10 died. Using the same proportion:

$$\frac{X}{1{,}000} = \frac{10}{6{,}098}$$

$$6{,}098\,X = 10{,}000$$

$$X = \frac{10{,}000}{6{,}098}$$

$X = 1.64$, the death rate per 1,000 at age 17

Age	No. of Policyholders at start of year	No. dying	Death rate per 1,000
16	5,844	9	1.54
17	6,098	10	1.64
18	6,433	11	_____
19	7,955	14	_____
20	9,551	17	_____

problems:

1. **Figure the death rate per 1,000 for the 18, 19 and 20-year old groups.**
2. **Twenty-five years ago, Academic Life's mortality experience for these age groups was as follows:**

Age	No. of Policyholders at start of year	No. dying	Death rate per 1,000
16	4,566	9	_____
17	5,017	11	_____
18	5,521	13	_____
19	6,392	15	_____
20	7,140	17	_____

(a) Compute the death rate per 1,000. (b) How do you account for the fact that the death rate per 1,000 now is considerably lower than it was 25 years ago? (c) How do you think this is reflected in the premium rates the company will charge?

3. **Academic Life had the following experience with its 16-year old policyholders over a five-year period:**

Year	No. of 16-year old Policyholders	No. dying	Death rate per 1,000
1958	5,141	8	_____
1959	5,576	9	_____
1960	5,392	8	_____
1961	5,676	10	_____
1962	5,793	9	_____

(a) Calculate the death rate per 1,000 for each year. (b) What is the *average* death rate per 1,000 for this five-year period? (c) Can you make a comparison between the various death rates you have just calculated, and the coin-tossing experiments you completed earlier? (d) What does the Law of Large Numbers mean when applied to yearly death rates?

how to construct a mortality table

Once the death rate per 1,000 for each age is known, it is not difficult to construct a mortality table for any group. You have already figured the death rate per 1,000, based on the experience of the Academic Life. Now, you can proceed to construct a mortality table for that company starting with 50,000 young people, age 16:

Age	No. of Policyholders at start of year	Death Rate per 1,000	No. Dying during year
16	50,000	1.54	77
17	49,923	1.64	82
18	49,841	1.71	_____
19	_____	1.76	_____
20	_____	1.78	_____

How do we conclude that 77 people die during the year? By taking the death rate per 1,000, as calculated on page 21. We can work it out as follows:

1.54 (death rate per 1,000 at age 16) × 50,000 = 77

The original group of 50,000 is now reduced by 77, leaving 49,923, age 17.

problem:

Using the given death rate at age 18 of 1.71, how many people may be expected to die of the 49,841 living at the start of the year? _____

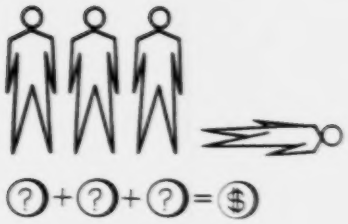
How many will be living at the start of the following year?

Continue the calculations to complete the table. (Note: round out the figures in each case, bearing in mind that we cannot have a fraction of a person dying.)

You have now constructed a portion of a mortality table for the Academic Life. You have based the table on a set of statistics which reflect the actual experience one life insurance company has had with a given set of policyholders.

using a mortality table

A person who buys an insurance policy agrees to pay the insurance company a certain amount of money annually. This specified sum of money is the *premium*. In order that a life insurance company may know how much the premium should be at each age, the company mathematicians — called actuaries — must base their calculations on a mortality table.



Suppose you were going to start a life insurance company. You have a group of 1,000 young people, all age 16, and you agree to insure each one for $1,000 for one year. The first thing you will want to know is how much money you will need to collect from each person in order to have enough on hand to pay the death claims of those who may die during the year. Where might you find how many of your group will die?

From your mortality table you learn that the death rate per 1,000 for 16-year olds is 1.54. That means that your company can expect to pay claims amounting to $1,540 at the end of the year:

$$1.54 \text{ deaths} \times \$1{,}000 = \$1{,}540$$

(In actual practice, the amount a company would have to pay in a particular year would not be exactly $1,540, but over a period of several years, the company could expect the average amount per year paid in claims on a group of 1,000 16-year old policyholders to be $1,540.)

If your company must be prepared to pay out $1,540 in death benefits at the end of the year, how much will you have to collect from each member of your group, assuming there are no expenses involved in operating the company?

$$\$1{,}540 \div 1{,}000 = \$1.54$$

problems:

1. **Look again at the mortality table you constructed for the Academic Life. You will note that again the death rate per 1,000 for a group of 16-year olds is 1.54. The Academic Life has a group of 5,000 policyholders, all age 16. How many death claims must the company be prepared to pay?**

   $1.54 \times 5 = X$, number of deaths in a group of 5,000 16-year olds.

   Each policyholder is insured for $1,000 for one year. How much will be paid out in claims during the year?

   $X \times \$1{,}000 = ?$

   What will each member of the group have to pay in order to meet the death claims? _____

2. **If the Academic Life has a group of 6,500 policyholders, all age 18, how many death claims must they be prepared to pay?** _____

   How much will be paid out as death claims during the year, assuming each policyholder is insured for $1,000? _____

   What will each person pay as a premium? _____

chapter six The matter of interest

The premium calculations you have just completed have been based solely on the death rate per 1,000 as indicated in tables of mortality. We have operated on the assumption that the premiums were collected at the start of the year, and that the money we had on hand was kept in a box in a safe, or that the company treasurer carried it around in his pocket. But we know better than this.

The money was collected and invested and earned *interest*. And the interest that is earned reduces the amount of money each policyholder must pay for his insurance.

Now we know of two essential factors to consider in calculating premiums: the *death rate* and the *interest rate*. Before we figure any further premium rates, it will be helpful if we review what we know about *compound interest*.

problems:

1. In the paragraph explaining compound interest, the calculation showed that at the end of the third year, the balance in the bank would come to $115.76. Show how the same answer can be obtained by consulting the table on page 25 and multiplying by 100.
2. If you deposit $50 in a bank that pays 5% interest, compounded annually, how much will be in your account at the end of 15 years? At the end of 25 years?
3. Make a table similar to the one on page 25 showing what happens to $1 invested at 5½% interest, compounded annually, for 1 2, 3, 4 and 5 years. Use your results to complete the following table. Does your entry for the fifth year check with one given here?
4. If you deposit $60 in a bank that pays 5% interest, compounded annually, how much will your account be at the end of 10 years? _____ At the end of 20 years? _____
5. What is the difference in dollars over the 10-year period if your bank pays 5½% interest compounded annually instead of 5% on your $60 deposit? _____ What is the difference at the end of 20 years? _____
6. How much would you have at the end of six years if you could invest $150 at 5% interest, compounded annually, for three years, and then compounded annually at 5½% for the next three years? _____

The Amount of $1 at the End of a Number of Years, Invested at 5½%, Compounded Annually

End of Year	Amount
1	$_____
2	_____
3	_____
4	_____
5	1.3070
10	1.7081
15	2.2325
20	2.9178
25	3.8134

the present value

Suppose you want to have $100 one year from now. How much would you have to put in the bank now so that with interest, at 5%, you would have exactly $100 at the end of one year?

Look again at the table and you see that $1 in one year's time will equal $1.050. Your problem is just the reverse: to find how much you need *now* in order to have $1 at the end of one year. You can represent this by means of this proportion:

$$\frac{X}{\$1.00} = \frac{\$1.00}{\$1.050}\ ;\ \frac{1.00}{\$1.050} = .9524$$

$$X = \$1.00 \times .9524 = \$.9524$$

We say that $.9524 (correct to four decimal places) is the *present value* of $1 due one year from now at 5% interest.

Thus .9524 × $100, or $95.24, is the amount you must save now in order to have $100 one year from now, if your money earns 5% interest annually.

present value formula

If we solve the compound interest formula for P, we obtain:

$$P = \frac{A}{(1+i)^n} \text{ or } P = A(1+i)^{-n} \text{ or } P = Av^n \text{ where } v = \frac{1}{1+i}$$

This is the formula for present value of an amount, A, due n years from now, at the interest rate of i per period, compounded annually. It shows us that the entries in a table of present values are the reciprocals of the corresponding entries in a compound interest table.

Present Value of $1 Due n Years From Now at 5% Interest, Compounded Annually	
End of n Years	Amount
1	$.9524
2	.9070
3	.8638
4	.8227
5	.7835
10	.6139
15	.4810
20	.3769
25	.2953

If you want to find out how much to set aside for a two-year period, you solve it in the same manner:

$$\frac{X}{\$1.00} = \frac{\$1.00}{\$1.1025}; \quad \frac{1.00}{1.1025} = .9070$$

$$X = \$1.00 \times .9070 = \$.9070$$

This is called the present value of $1 due two years from now at 5% interest, compounded annually. The table above shows how much money must be invested at 5% interest, compounded annually, in order to amount to exactly $1 n years from now.

From this table, you learn that in order to have $1, five years from now, you need save only 78.35¢ now, if your money can earn interest at 5% compounded annually. Or if you want $100 in five years, you need set aside $78.35 now.

problems:

1. **Jack Russell will need $1,800 at the end of five years. His bank pays 5% interest, compounded annually. How much will Jack have to deposit now in order to have the money he needs five years from now?** _____
2. **Using the table you devised for the amount of $1 at 5½% interest, compounded annually, complete the following table of present values of $1 invested at 5½% interest, compounded annually for 1, 2, 3, 4 and 5 years. Does your fifth year value check with the one on the following page?**

Present Value of $1 Due n Years From Now at 5½% Interest, Compounded Annually

End of Years *N*	Amount v^n
1	$____
2	____
3	____
4	____
5	.7651
10	.5854
15	.4479
20	.3427
25	.2622

3. If money earns 5% interest, compounded annually, what is the present value of $15,000 due 25 years from now? _____

4. Jerry Stone wants to have $5,000 15 years from now to finance a college education for his son. How much does he have to set aside now in order to have that money, assuming his money will earn 5½% interest, compounded annually? _____

5. Barbara Horlick will graduate from high school in two years. She wants $175 at that time so that she can take a class trip to Washington, D.C. Her grandfather has agreed that he will deposit enough money in a savings account now so that Barbara will have $175 at the end of two years. If the bank pays 5% interest, compounded annually, how much will her grandfather have to deposit now? _____

6. What is the present value at 5½% interest, compounded annually, of $3,000 due 5 years from now, plus $5,000 due 10 years from now? _____

Figuring the premium

chapter seven

Present Value of $1 Due n Years From Now at 4% Interest, Compounded Annually

End of Years *n*	Amount v^n
1	$.9615
2	.9246
3	.8890
4	.8548
5	.8219
10	.6756
15	.5553
20	.4564
25	.3751

Consider again the life insurance company organized to insure 1,000 16-year olds for $1,000 for one year. Recall that your company expects to pay claims amounting to $1,540. Assume that premiums are paid at the beginning of the year, and benefits are paid at the end of the year. Assume, too, that your company expects to earn 4% interest a year.

From the present value table on the left, you know that for every dollar due one year from now, you need set aside now only $.9615, assuming a 4% interest rate. Your life insurance company, therefore, will need to collect only $.9615 at the beginning of the year for every dollar it will pay out at the end of the year.

You know how much you'll need to pay out: $1,540.

You know how much you need now for each dollar to be paid out later: $.9615.

You are ready to figure the premium:

$$\$1{,}540 \times .9615 = \$1{,}480.71$$

$$\frac{\$1{,}480.71}{1{,}000 \text{ policyholders}} = \$1.4807 \text{ per policyholder, or } \$1.48, \text{ rounded to the nearest cent.}$$

The premium each policyholder will have to pay is $1.48. Let's check out the answer, just to be sure our calculation is correct:

$\$1.48$ = premium to be paid by 1,000 16-year olds

$\$1.48 \times 1{,}000$ = $1,480, the total amount to be invested at 4%

$\$1{,}480 \times 4\%$ = $59.20, the interest earned on the invested dollars

$\$1{,}480 + \$59.20 = \$1{,}539.20$ with which to pay claims.

The difference of $.80 between the actual amount needed and amount obtained is the result of having rounded the premium to the nearest cent, and because the present value table is given to only four decimal places.

formula for one-year term premium

If we use q_x to indicate the death rate per 1,000 at age x, we can find a simple formula for the net premium, p_x, per $1,000 of one-year term insurance.

If we let the number of persons insured be 1,000, the amount needed to pay estimated death claims is $1,000 q_x. The present value of these claims is:

$$\frac{\$1{,}000\, q_x}{1+i}$$

Since 1,000 persons pay premiums:

$$p_x = \frac{\$1 \times q_x}{1+i}$$

The premium just calculated is what the company would call the *net* premium because it does not make any provision for the costs of running the company, commissions to salesmen and other expenses. These costs are estimated in advance, and the "expense factor" becomes the third element in the calculation of a premium. The final premium, figured on the basis of the death rate, interest rate, and expense rate, is the *gross* premium, which is the true amount paid to the company. For our purposes, we will calculate only net premiums in the problems to follow.

problems:

1. Using the death rates in the C.S.O. Table, and a 4% interest rate, figure the net premium for $1,000 of life insurance for one year for the following age groups:

1,000 persons, age 30 _____

age 40 _____

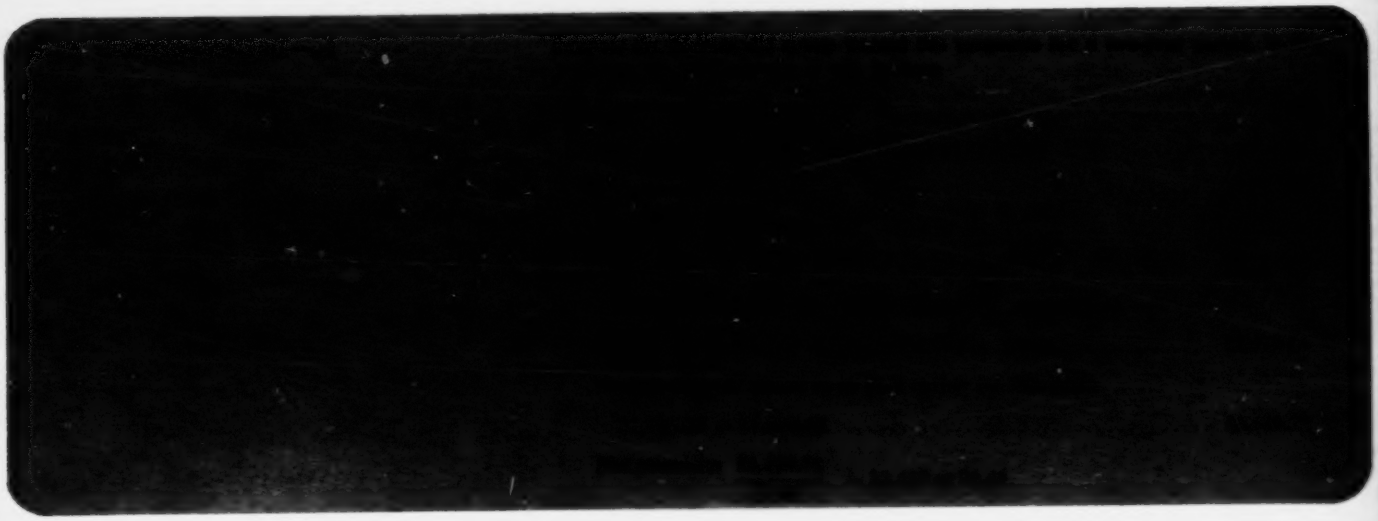
age 50 _____

age 65 _____

2. Suppose the 20-year old group, in Class Demonstration I, decided that each member should be insured for $3,000 for one year. What would the premium be? _____ What would the premium be for $5,000 of life insurance at age 30? _____ $7,500 of life insurance at age 40? _____

3. Draw a graph showing the net premiums for $1,000 of life insurance at ages: 16, 20, 30, 40, 50 and 65.

4. Show that by dividing the death rate per 1,000 at age 20 by 1.0400 you obtain the net premium as in Class Demonstration I. (Note: Your answer will be .0001 more than in Class Demonstration I because the interest table value used in Class Demonstration I is rounded.)



In this calculation, we have arrived at the net premium for a two-year policy, but the premium is payable only one time – at the start of the first year. Such policies are called "single premium policies."

problems:

1. Calculate the net premium for a two-year policy for a group of 1,000 all age 30, with interest at 4%.

2. In Demonstration II, we calculated the premium for a two-year policy at age 20. Extend this to provide the premium rate for a three-year policy.

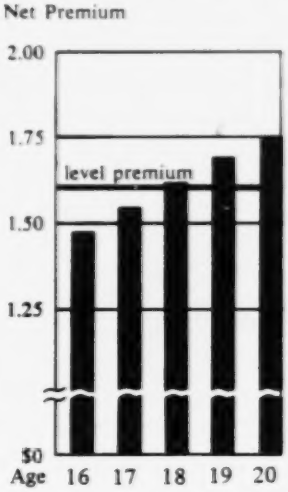
chapter eight The annual premium

The two-year policy we computed in the previous chapter could do only one thing: it provided protection for one or two years. At the end of the period, protection ceased. However, people usually need protection for a much longer period of time. They can buy it one year at a time, paying more for it as each year passes. Recall the graph you designed, showing what happens to death rates as one gets older. Premium rates for this insurance increase every year with the increasing death rate, which for most persons makes a one-year term plan impractical. People usually want their insurance to continue, and they want to pay the same amount annually.

To meet this need, life insurance companies have devised a method of premium calculation that eliminates yearly increases. This method is called the *level premium system.* It is so named because it spreads the outlay for insurance evenly over the number of years the life insurance policy is in force. In other words, the annual premium remains the same, or level, even though the risk of dying increases with each year's passage.

Age (x)	Number Living (l_x)	Number Dying (d_x)
16	9,728,950	14,983
17	9,713,967	15,737
18	9,698,230	16,390
19	9,681,840	16,846
20	9,664,994	17,300
21	9,647,694	17,655

Net annual premiums for $1,000 of life insurance, calculated one year at a time, and showing the level premium at age 15 and extending over the same number of years.



The chart at right pictures for us how the two systems, "level premium" and "one-year-at-a-time," compare. You will notice how the vertical bars step up each year as the number of deaths increase. The horizontal bar shows no step up, but it does show that the premium rate in the first years is greater than the term rate, and that in the later years it becomes less. The extra dollars paid in the early years are used to pay toward the insurance in the later years when the numbers of deaths increase sharply. This is the level premium system, and the elements that go into calculating the level premium are the company's mortality experience, anticipated income from its investments, and the company's administrative expenses. In discussing premium calculations in this chapter, we will concentrate only on mortality and interest rates that are applicable to the various types of insurance discussed.

We shall use the following notation to compute the net annual premium:

$$1000\ P\ {}^{1}_{16:\overline{5}|}$$

This notation represents the premium on a $1000 policy issued to a 16-year old, with the $1000 benefit payable if the person dies before the expiration of 5 years. The calculation is based on an equation stating that the present value of the net annual premium is equal to the present value of the benefits. We write:

Present Value of Net Annual Premiums = Present Value of Benefits

If l_x represents the number living, and d_x the number dying, at age x, and v^n the present value of $1 due n years from now at interest compounded annually, then this equation becomes

$$1000\ P\ {}^{1}_{16:\overline{5}|} \quad (l_{16} + l_{17}v + l_{18}v^2 + l_{19}v^3 + l_{20}v^4) =$$

$$(1000)\ (d_{16}v + d_{17}v^2 + d_{18}v^3 + d_{19}v^4 + d_{20}v^5)$$

We note that premiums are assumed to be paid at the beginning of each year of age, death benefits to be paid at the end of the year of death.

Since the members of this equation are so lengthy, let us compute them separately, using L for the left member and R for the right member. The values of l_x and d_x are obtained from the part of the CSO mortality table just shown; the values of v^n are valued at 4% interest and taken from the table on page 28, Chapter 7. Substituting these values in the left member of the equation, we have:

$$L = 1000\ P\ {}^{1}_{16:\overline{5}|} \cdot [(9{,}728{,}950) + (9{,}713{,}967 \times .9615) + (9{,}698{,}230 \times .9246) + (9{,}681{,}840 \times .8890) + (9{,}664{,}994 \times .8548)]$$

$$L = 1000\ P\ {}^{1}_{16:\overline{5}|} \cdot [(9{,}728{,}950 + 9{,}339{,}979.27 + 8{,}966{,}983.46 + 8{,}607{,}155.76 + 8{,}261{,}636.87)]$$

$$L = 1000\ P\ {}^{1}_{16:\overline{5}|} \cdot (44{,}904{,}705.36)$$

Next we substitute in the right member:

$$R = 1000 \cdot (14{,}983 \times .9615 + 15{,}737 \times .9246 + 16{,}390 \times .8890 + 16{,}846 \times .8548 + 17{,}300 \times .8219)$$

$$R = 1000 \cdot (14{,}406.15 + 14{,}550.43 + 14{,}570.71 + 14{,}399.96 + 14{,}218.87)$$

$$R = 1000 \cdot (72{,}146.12)$$

$$R = 72{,}146{,}120$$

Now we equate the values for the left and right members of our equation:

$$1000\ P_{\overline{16:5}|}^{1} \cdot (44{,}904{,}705.36) = 72{,}146{,}120$$

Solving: $1000\ P_{\overline{16:5}|}^{1} = \frac{72{,}146.120}{44{,}904{,}705}$

$$1000\ P_{\overline{16:5}|}^{1} = 1.61$$

Thus the premium would be $1.61 each year for five years.

A simpler method of calculation

The calculation of the premium just made is, as you can see, a complicated and time consuming process. To simplify the calculations, actuaries have developed a table called a Commutation Table. A portion of such a table is given here.

Age (x)	D_x	N_x	M_x
16	5,194,365.9	115,325,945	758,752.665
18	4,787,319.1	105,144,688	743,292.568
21	4,233,732.3	91,350,980	720,233.064
23	3,899,893.0	83,053,801	705,516.072
26	3,447,162.9	71,819,116	684,889.197
28	3,174,519.4	65,063,870	672,062.893

It is not appropriate here to go into detail about this table. For our purposes it is sufficient to say that D_x and N_x have to do with the number of persons living, and M_x with the number of persons dying. An interest factor is also included and, the above table assumes 4% interest. This table can be used to calculate the premiums for various types of life insurance. For example, the

premium for a $1000 five-year term policy for a 16 year old would be obtained as follows:

$$1000\,P_{\overline{16:5}|}^{1} = 1000 \cdot \frac{M_{16} - M_{21}}{N_{16} - N_{21}}$$

$$= 1000 \cdot \frac{758{,}752.665 - 720{,}233.064}{115{,}325{,}945 - 91{,}350{,}980}$$

$$= 1000 \cdot \frac{38{,}519.601}{23{,}974{,}965}$$

$$= 1000\ (.00161)$$

$$1000\,P_{\overline{16:5}|}^{1} = 1.61$$

Thus the annual premium is $1.61, the same answer as obtained earlier, but this time by a much simpler computation.

endowment policies

An endowment policy provides life insurance protection for a certain number of years; at the end of the period the protection ceases, and the face value of the policy is paid to the policyholder. For example, if an individual with a five-year endowment policy dies within five years, the proceeds of the policy are paid to the beneficiary named in the policy. But if the policyholder lives to the end of five years, the face value of the policy is paid to him or her.

The commutation table can be used to calculate the amount of premium on a 5-year endowment policy for $1000 issued at age 16 in this way:

$$1000\,P_{\overline{16:5}|} = \frac{1000\,(M_{16} - M_{21} + D_{21})}{N_{16} - N_{21}}$$

$$1000\,P_{\overline{16:5}|} = \frac{1000\,(758{,}752.665 - 720{,}233.064 + 4{,}233{,}732.3)}{115{,}325{,}945 - 91{,}350{,}980}$$

$$= \frac{1000\,(4{,}272{,}251.9)}{23{,}974{,}965}$$

$$= 1000\ (.17820)$$

$$1000\,P_{\overline{16:5}|} = 178.20$$

Thus the net annual premium would be $178.20.

straight life policy

For a straight life policy, also called whole life insurance, the policyholder pays the same annual premium throughout the lifetime of the insured; the

benefit is payable at the death of the insured. Straight life policies are the kind most families own because they are the least expensive of all non-term life insurance, and because they are flexible and can be used for cash values and added retirement income.

The formula for the annual premium on a $1000 straight life policy insuring a person aged 16 follows:

$$1000\,P_{16} = \frac{1000\,M_{16}}{N_{16}}$$

A variation of straight life is the limited payment policy. This provides lifetime protection, but premiums are payable only for a stated number of years. You may hear of a "20-pay life" or a "life paid up at 65" policy, for example. The premiums are higher than for a straight life policy. To solve for the premium for 5-year Annual Payment Whole Life Insurance you would use the following formula:

$$1000 \cdot {}_5P_{16} = \frac{1000\ M_{16}}{N_{16} - N_{21}}$$

This premium is payable for 5 years and the benefit is payable when the insured dies.

problems

1. Using the formula given for a straight life policy, find the annual premium on such a policy for a person aged 16. (The values to be substituted are in the Commutation Table.)

2. Do the same problem as number 1, but for age 28. How do you account for the difference in the answers to these two problems?

3. Using the formula for limited payment life, find the premium for a 5-year annual payment whole life policy for a person aged 16. Explain the difference between your answer in this problem and the answer in problem 1.

4. Adjust the formula for a 5-year endowment policy so that it applies to a $1000 5-year policy issued at age 18. Complete the computation, using values from the Commutation Table. How does your answer differ from that when the age is 16? Why?

5. Adjust the formula for a 5-year endowment policy so that it applies to a $1000 10-year endowment policy issued at age 16. Complete the computation.

6. Use your answer to problem 5 to find the total of the premiums paid in 10 years; use the answer worked out in the text for a 5-year policy to find the total of the premiums paid. Do you see any reason why one of these totals is larger than the other?

Education and Community Services
American Council of Life Insurance
1850 K Street, N.W.
Washington, D.C. 20006

LITHO IN U.S.A.

9/78

176